QMath13, 10th October 2016

Classification theory of topological insulators with Clifford algebras and its application to interacting fermions

Takahiro Morimoto

UC Berkeley

Berkeley UNIVERSITY OF CALIFORNIA

GORDON AND BETTY MOORE FOUNDATION

Collaborators

- Akira Furusaki (RIKEN)

- Christopher Mudry (PSI)

Morimoto, Furusaki, PRB 88, 125129 (2013)
Morimoto, Furusaki, Mudry, Phys. Rev. B 91, 235111 (2015)

Plan of this talk

- Introduction
  - Topological insulators and superconductors
  - Ten fold way classification
- Classification theory of topological insulators
  - Massive Dirac Hamiltonian
  - Clifford algebras and classifying spaces
  - Application to topological crystalline insulators
- Breakdown of ten fold way classification with interactions
  - Dynamical mass terms
  - Nonlinear sigma model

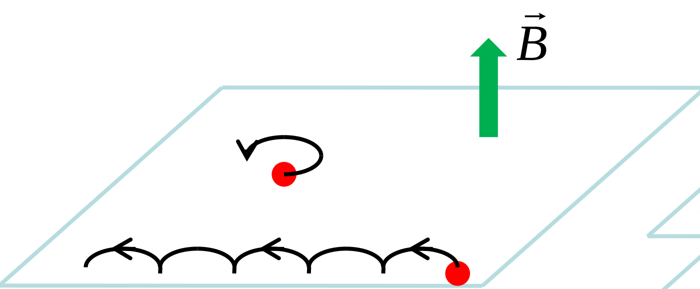
Topological insulator/superconductor is :

- A system of non-interacting fermions with a band gap.
  - Band insulator
  - Superconductor with a full gap (BdG equation)

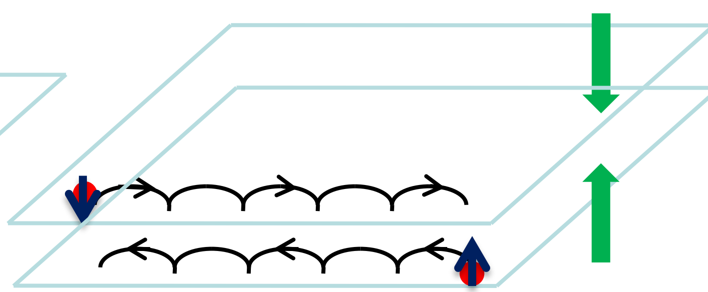
- Characterized by a non-trivial topological number (Z or Z2).

- Accompanied with a gapless surface state.

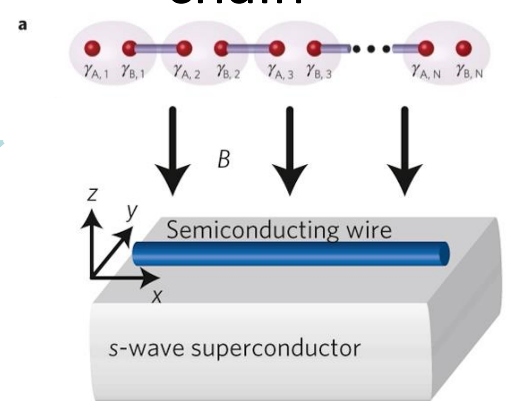
IQHE



QSHE



Majorana chain

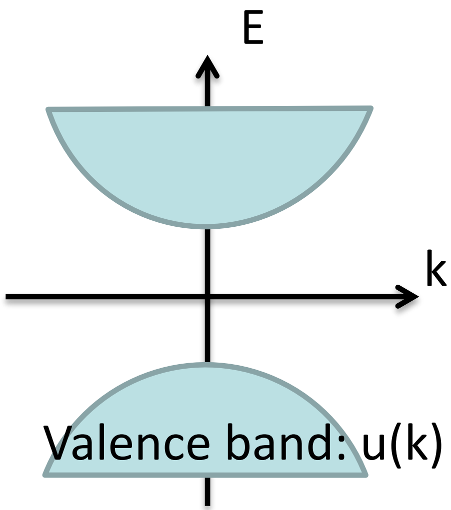


Topology of energy band

Energy band $k \rightarrow E(k)$

Bloch wavefunction $k \rightarrow |u(k)>$

Insulator



Brillouin zone

k

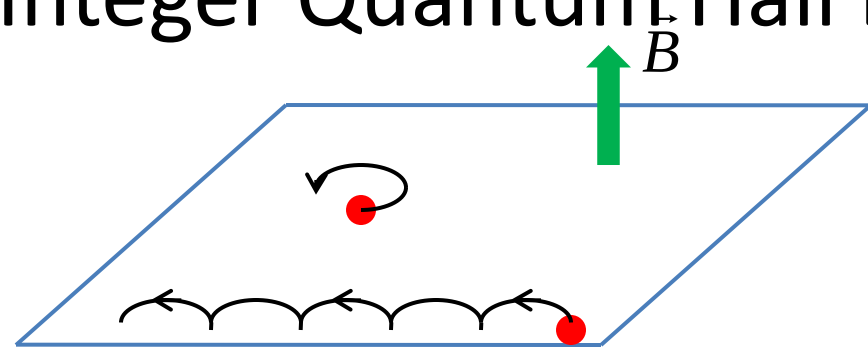
Valence wavefunctions

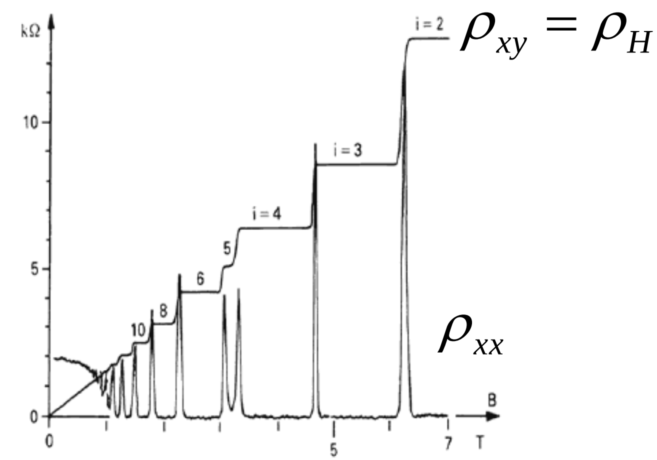
|u(k) >

Non trivial way that the Brillouin zone wraps the space of valence wavefucntion.

= Topological insulators and superconductors

Integer Quantum Hall Effect





TKNN number (Thouless-Kohmoto-Nightingale-den Nijs)

$$\sigma_{xy} = -\frac{e^2}{h} C$$

TKNN (1982); Kohmoto (1985)

1st Chern number

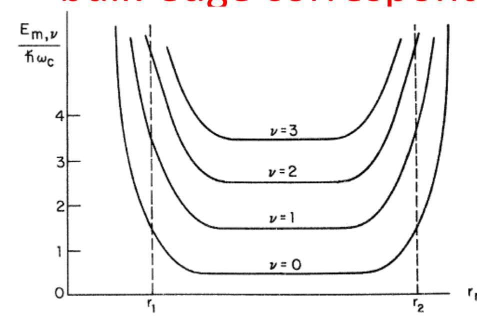
integer valued

$$C = \frac{1}{2\pi i} \int_{\text{filled band}} d^2k \; \vec{\nabla}_k \times \vec{A}(k_x, k_y)$$ = number of edge modes crossing E_F

bulk-edge correspondence

$$\vec{A}(k_x, k_y) = \langle \vec{k} | \vec{\nabla}_k | \vec{k} \rangle$$ Berry connection

$$\vec{\nabla}_k = (\partial_{k_x}, \partial_{k_y})$$



- ☐ Systematic understanding of topological phases?
- ☐ Relationships to the symmetry and the dimensionality?

- ● A system of non-interacting fermion is classified into 10 Altland-Zirnbauer classes
- ● 5 classes of non-trivial TI/TSC for each dimension

Table of topological insulators/superconductors

		TRS	PHS	CS	d=1	d=2	d=3
Standard (Wigner-Dyson)	A (unitary)	0	0	0	--	Z (IQHE)	--
	AI (orthogonal)	+1	0	0	--	--	--
	AII (symplectic)	-1	0	0	--	Z_2 (QSHE)	Z_2 (Z_2TPI)
Chiral	AIII (chiral unitary)	0	0	1	Z	--	Z
	BDI (chiral orthogonal)	+1	+1	1	Z (polyacetylene (SSH))	--	--
	CII (chiral symplectic)	-1	-1	1	Z	--	Z_2
BdG	D (p-wave SC)	0	+1	0	Z_2 (p SC)	Z (p+ip SC)	--
	C (d-wave SC)	0	-1	0	--	Z (d+id SC)	--
	DIII (p-wave TRS SC)	-1	+1	1	Z_2	Z_2	Z (^{3}He-B)
	CI (d-wave TRS SC)	+1	-1	1	--	--	Z

Schnyder, Ryu, Furusaki, and Ludwig, PRB (2008)

Ten Altland-Zirnbauer symmetry classes

Bilinear Hamiltonian: $$\mathcal{H} = \Psi^\dagger H \Psi$$

Fully block-diagonalized Hamiltonian matrix

Three generic symmetries:

-- Time-reversal symmetry ← Without B or magnetization

$$THT^{-1} = H \qquad TiT^{-1} = -i$$

-- Particle hole symmetry ← BdG equation, (superconductors)

$$CHC^{-1} = -H \qquad CiC^{-1} = -i$$

-- Chiral symmetry ← Sublattice symmetry, combination of TC

$$\Gamma H = -H\Gamma \qquad \Gamma i \Gamma^{-1} = i$$

class	T	C	Γ
A	0	0	0
AIII	0	0	1
AI	+1	0	0
BDI	+1	+1	1
D	0	+1	0
DIII	−1	+1	1
AII	−1	0	0
CII	−1	−1	1
C	0	−1	0
CI	+1	−1	1

Derivation of the topological periodic table: Dirac Hamiltonian and topological phase

$$H = \sum_{i=1}^{d} k_i \gamma_i + m\gamma_0$$

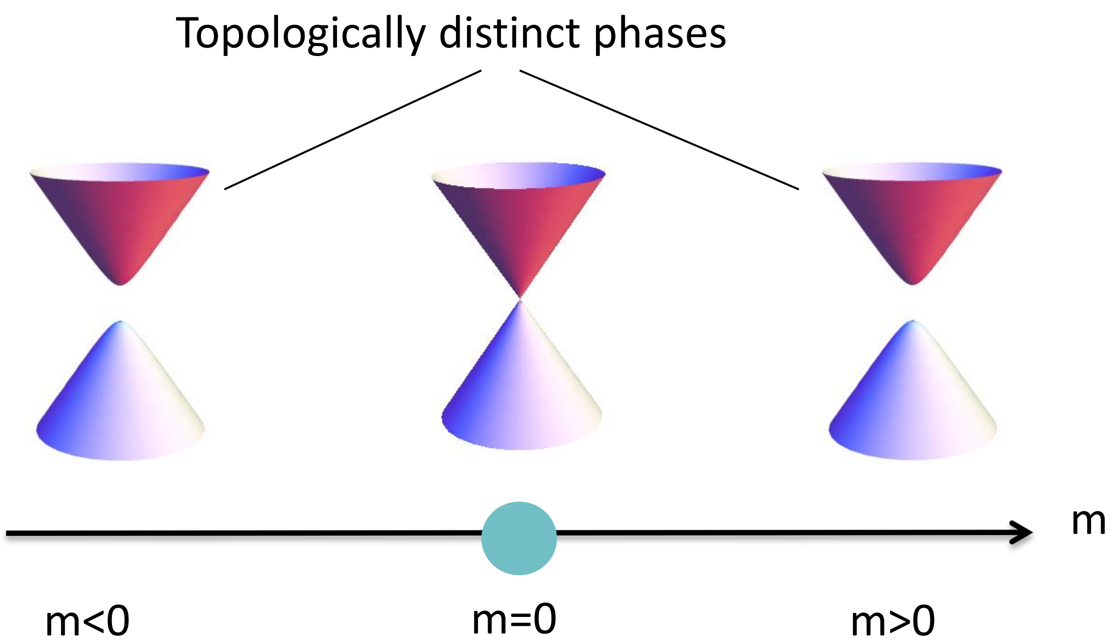
(Assumption: Any gapped Hamiltonian can be deformed into the Dirac form)

Gapped phase

⇕

Massive Dirac Hamiltonian

- If Dirac mass $\gamma 0$ is unique,



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Massive Dirac Hamiltonian

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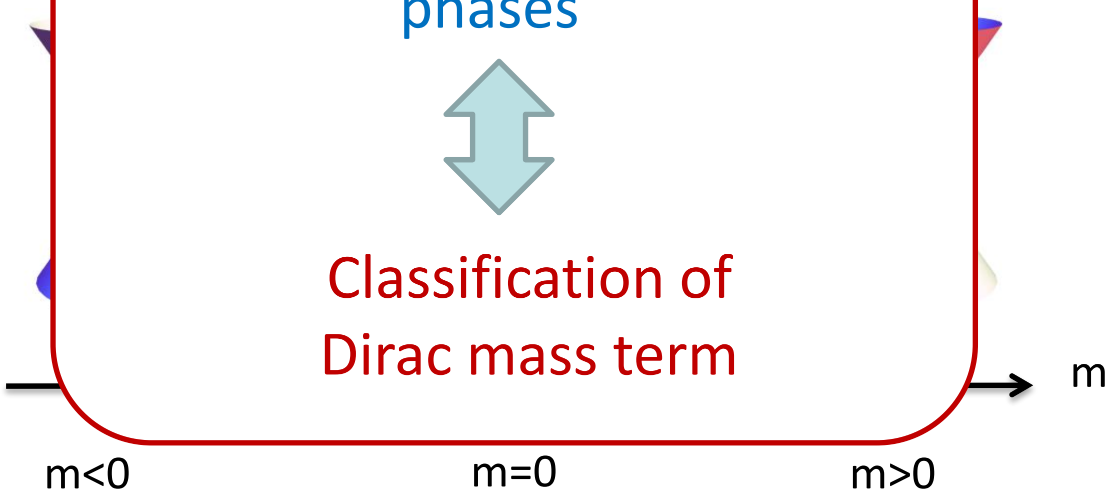
Classification of topological phases

⇕

Classification of Dirac mass term



Clifford algebras

Complex Clifford algebra: Cl_n

n generators: e_i $\{e_i, e_j\} = 2\delta_{i,j}.$

Real Clifford algebra: $Cl_{p,q}$

p+q generators: e_i

$$\{e_i, e_j\} = 0 \quad i \neq j$$

$$e_i^2 = \begin{cases} -1 & (1 \leq i \leq p) \\ +1 & (p+1 \leq i \leq p+q) \end{cases}$$

$2^{(p+q)}$ –dim real vector space spanned by bases of combinations of e_i's

Clifford algebra of real symmetry classes

$$H(k) = \sum_{i=1}^{d} k_i \gamma_i + m\gamma_0$$

Time-reversal symmetry T $\{T, \gamma_i\} = [T, \gamma_0] = 0, \quad T^2 = \pm 1$

Particle-hole symmetry C $[C, \gamma_i] = \{C, \gamma_0\} = 0, \quad C^2 = \pm 1$

T and C are antiunitary $\{T, i\} = \{C, i\} = 0$

J represents for "i", $J^2 = -1$

(i) T only (AI & AII): $e_0 = J\gamma_0$, $e_1 = T$, $e_2 = TJ$, $e_3 = \gamma_1$, … , $e_{2+d} = \gamma_d$

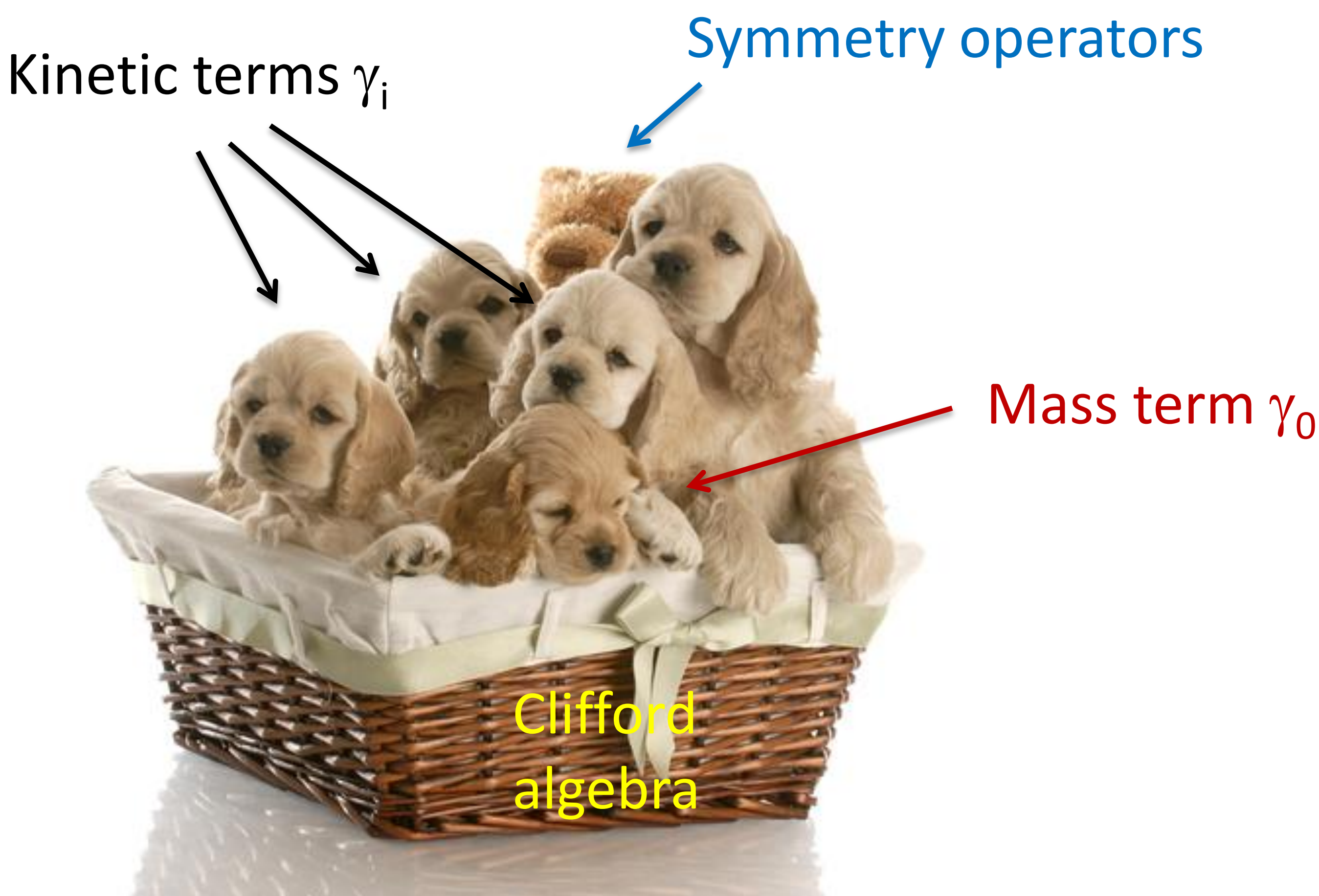
(ii) C only (D & C): $e_0 = \gamma_0$, $e_1 = C$, $e_2 = CJ$, $e_3 = J\gamma_1$, … , $e_{2+d} = J\gamma_d$

(iii) T and C (BDI, DIII, CII & CI): $e_0 = \gamma_0$, $e_1 = C$, $e_2 = CJ$, $e_3 = TCJ$, $e_4 = J\gamma_1$, … , $e_{3+d} = J\gamma_d$

Symmetry constraints for class D:

$$\{\gamma_i, \gamma_j\} = 2\delta_{i,j} \quad [C, \gamma_i] = \{C, \gamma_0\} = 0, \quad \{C, i\} = 0$$

$$e_0=\gamma_0, \quad e_1=C, \quad e_2=iC, \quad e_3=i\gamma_1, \ldots , e_{2+d}=i\gamma_d$$

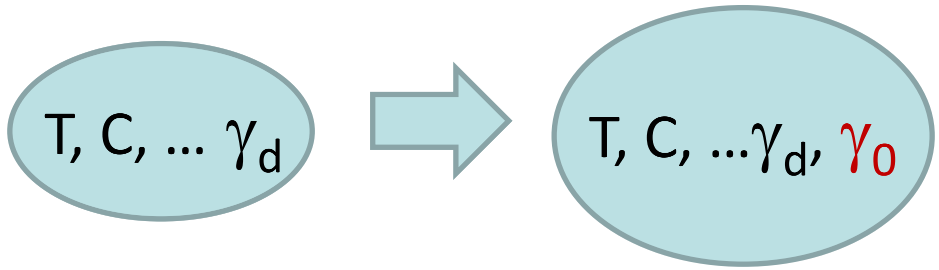


Classification of topological insulators

= Classification of Dirac mass term

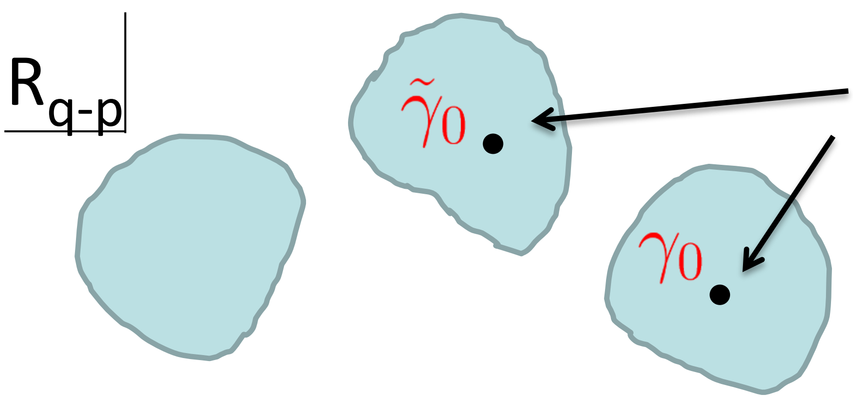
(i) We consider extension of Clifford algebra without γ_0 into algebra with γ_0

$$Cl_{p,q} \to Cl_{p,q+1}$$



(ii) All the possible extensions ("classifying space") = space of γ_0

R_{q-p}



γ_0 and $\tilde{\gamma}_0$ in disconnected parts

➔ Distinct topological phases

(iii) Topological classification = $\pi_0(R_{q-p})$

Kitaev 09

				(i)	(ii)	(iii)
class	T	C	Γ	extension	V	$\pi_0(V_{d=0})$
A	0	0	0	$Cl_d \rightarrow Cl_{d+1}$	C_{0+d}	$\mathbb{Z}$
AIII	0	0	1	$Cl_{d+1} \rightarrow Cl_{d+2}$	C_{1+d}	0
AI	$+1$	0	0	$Cl_{0,d+2} \rightarrow Cl_{1,d+2}$	R_{0-d}	$\mathbb{Z}$
BDI	$+1$	$+1$	1	$Cl_{d+1,2} \rightarrow Cl_{d+1,3}$	R_{1-d}	$\mathbb{Z}_2$
D	0	$+1$	0	$Cl_{d,2} \rightarrow Cl_{d,3}$	R_{2-d}	$\mathbb{Z}_2$
DIII	-1	$+1$	1	$Cl_{d,3} \rightarrow Cl_{d,4}$	R_{3-d}	0
AII	-1	0	0	$Cl_{2,d} \rightarrow Cl_{3,d}$	R_{4-d}	$\mathbb{Z}$
CII	-1	-1	1	$Cl_{d+3,0} \rightarrow Cl_{d+3,1}$	R_{5-d}	0
C	0	-1	0	$Cl_{d+2,0} \rightarrow Cl_{d+2,1}$	R_{6-d}	0
CI	$+1$	-1	1	$Cl_{d+2,1} \rightarrow Cl_{d+2,2}$	R_{7-d}	0

Bott periodicity

Complex and real K-theory

Topological periodic table from Clifford algebra

$\pi_0(C_{q-d})$

	d												
Cartan	0	1	2	3	4	5	6	7	8	9	10	11	...
Complex case:													
A	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	
AIII	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	
Real case:													
AI	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	...
BDI	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	...
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	
DIII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	
AII	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	
CII	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	...
C	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	...
CI	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	...

period $d = 2$

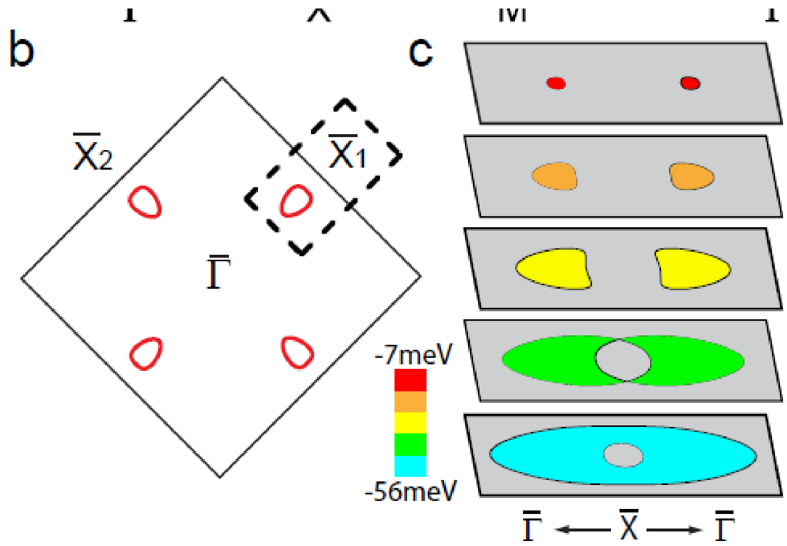
period $d = 8$

$\pi_0(R_{q-d})$

Topological crystalline insulator

- Topological insulator with
  time-reversal + Reflection symmetry
  ($Z_2 \rightarrow Z$)
  -- SnTe compounds
  with even # of surface Dirac cones

Experiments （SnTe/PbTe）:
Tanaka et al., Nat. Phys. 2012
Xu et al., Nat. Commun. 2012
Dziawa et al., Nat. Mat. 2012

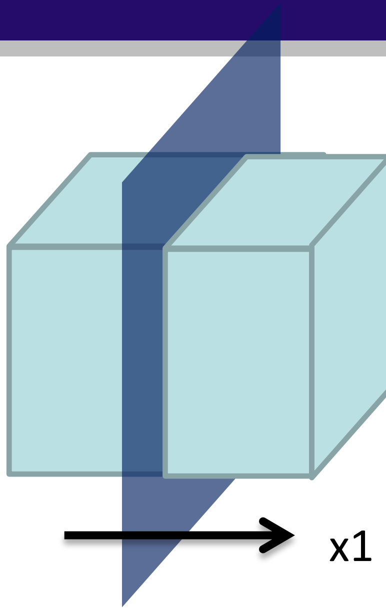


Hsieh et al. Nat. Commun. 2012

Reflection symmetry

Reflection symmetry with 1-direction

$$R^{-1}H(-k_1, k_i)R = H(k_1, k_i)$$



Dirac Hamiltonian $H = m\gamma_0 + \sum_{i=1}^{d} k_i \gamma_i,$

$$\{R, \gamma_1\} = 0, \quad [R, \gamma_i] = 0 \quad (i \neq 1)$$

R^{η_T, η_C} gives an additional chiral symmetry M

$$M = i\gamma_1 R^{\eta_T, \eta_C}$$ anti-commutes with all γ_i's

→ New generator of Clifford algebra

Topological periodic table with a reflection symmetry

$$R^{\eta_T} T = \eta_T T R^{\eta_T} \quad \text{and} \quad R^{\eta_C} C = \eta_C C R^{\eta_C}$$

Reflection	Class	C_q or R_q	$d=0$	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$
R	A	C_1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
R^{+}	AIII	C_0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
R^{-}	AIII	C_1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
	AI	R_1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
	BDI	R_2	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
R^{+}	D	R_3	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
	DIII	R_4	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
R^{++}	AII	R_5	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
	CII	R_6	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
	C	R_7	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
	CI	R_0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
	AI	R_7	0	0	0	$\mathbb{Z}$	0	“$\mathbb{Z}_2$”	$\mathbb{Z}_2$	$\mathbb{Z}$
	BDI	R_0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	“$\mathbb{Z}_2$”	$\mathbb{Z}_2$
R^{-}	D	R_1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	“$\mathbb{Z}_2$”
	DIII	R_2	“$\mathbb{Z}_2$”	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
R^{--}	AII	R_3	0	“$\mathbb{Z}_2$”	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
	CII	R_4	$\mathbb{Z}$	0	“$\mathbb{Z}_2$”	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
	C	R_5	0	$\mathbb{Z}$	0	“$\mathbb{Z}_2$”	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
	CI	R_6	0	0	$\mathbb{Z}$	0	“$\mathbb{Z}_2$”	$\mathbb{Z}_2$	$\mathbb{Z}$	0
R^{+-}	BDI	R_1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
R^{-+}	DIII	R_3	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
R^{+-}	CII	R_5	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
R^{-+}	CI	R_7	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
R^{-+}	BDI, CII	C_1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
R^{+-}	DIII, CI	C_1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$

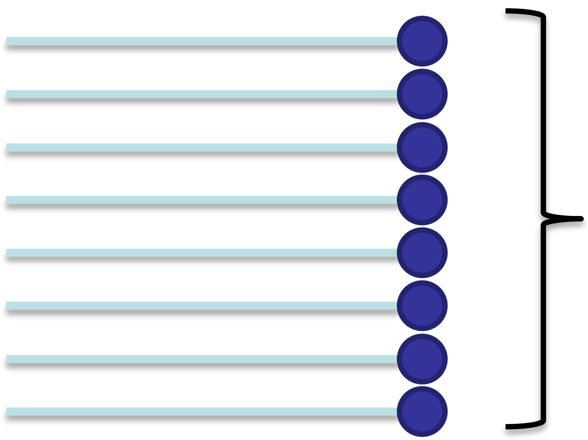
SnTe
(class AII+R- d=3)

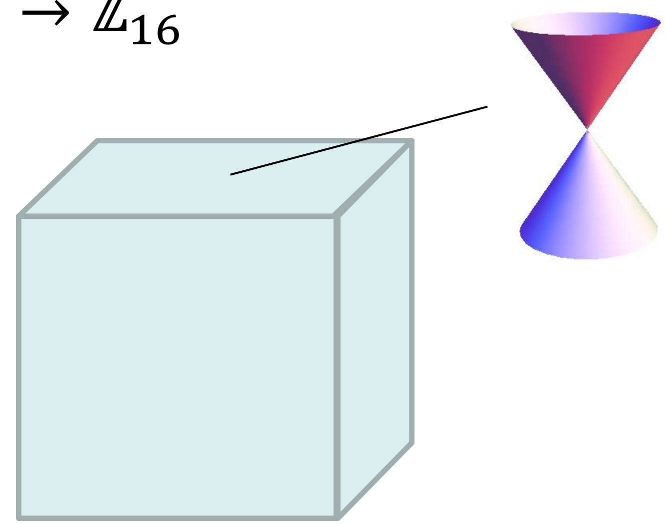
Interaction effects on topological insulators and superconductors

Breakdown of non-interacting topological phases labeled by Z with interactions

Time-reversal symmetric Majorana chain (1D class BDI)
$\mathbb{Z} \to \mathbb{Z}_8$

Time-reversal symmetric 3D topological SC (3D class DIII)
$\mathbb{Z} \to \mathbb{Z}_{16}$





8 Majorana zero modes at the boundary can be gapped without breaking TRS.

16 Dirac surface fermions can be gapped without breaking TRS.

Fidkowski and Kitaev, PRB (2010), PRB (2011)

Kitaev (2011), Fidkowski etal. PRX (2013), Metlitski, Kane & Fisher. (2014), ….

Aim: Systematic study of the breakdown of Z classification

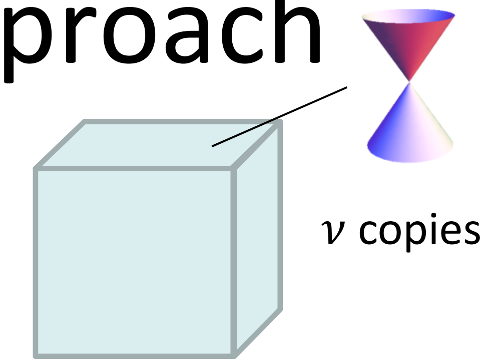
- Stability analysis of boundary gapless states against interactions in any dimension and any symmetry class
  ← Nonlinear sigma model

- Applications to the tenfold way and other topological phases (topological crystalline insulators)

Result:

Class	T	C	Γ_5	V_d	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$	$d=8$
A	0	0	0	C_{0+d}	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	0	0	1	C_{1+d}	$\mathbb{Z}_4$	0	$\mathbb{Z}_8$	0	$\mathbb{Z}_{16}$	0	$\mathbb{Z}_{32}$	0
AI	+1	0	0	R_{0-d}	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	+1	+1	1	R_{1-d}	$\mathbb{Z}_8$	0	0	0	$\mathbb{Z}_{16}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	0	+1	0	R_{2-d}	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
DIII	−1	+1	1	R_{3-d}	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{16}$	0	0	0	$\mathbb{Z}_{32}$	0
AII	−1	0	0	R_{4-d}	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
CII	−1	−1	1	R_{5-d}	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{16}$	0	0	0
C	0	−1	0	R_{6-d}	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
CI	+1	−1	1	R_{7-d}	0	0	$\mathbb{Z}_4$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{32}$	0

Nonlinear sigma model approach



ν copies of gapless boundary states

<u>Boundary massless Dirac fermions + quartic interactions</u>

$$\mathcal{L}_{\mathrm{bd}} := \Psi^\dagger \left(\partial_\tau + \mathcal{H}_{\mathrm{bd}}^{(0)} \right) \Psi + \lambda \sum_{\{\beta\}} \left(\Psi^\dagger \beta \, \Psi \right)^2 .$$

$$\mathcal{H}_{\mathrm{bd}}^{(0)} = \sum_{i=1,\ldots,d-1} (-i)\partial_i \alpha_i \qquad \{\beta_1, \beta_2, \dots, \beta_N\}$$

α, β: anti-commuting gamma matrices

α respects symmetries, β can be odd under some symmetry operations.

<u>Hubbard-Stratonovich transformtation</u>

$$\mathcal{L}'_{\mathrm{bd}} := \Psi^\dagger \left(\partial_\tau + \mathcal{H}_{\mathrm{bd}}^{(\mathrm{dyn})} \right) \Psi + \frac{1}{\lambda} \sum_{\{\beta\}} \phi_\beta^2 ,$$

$$\mathcal{H}_{\mathrm{bd}}^{(\mathrm{dyn})}(\tau, \boldsymbol{x}) := \mathcal{H}_{\mathrm{bd}}^{(0)}(\boldsymbol{x}) + \sum_{\{\beta\}} 2\mathrm{i}\, \beta \, \phi_\beta(\tau, \boldsymbol{x}),$$

Dynamical Dirac masses

cf. You and Xu, PRB (2014), Kitaev's talk (2015)

Integration of fermions

$$\mathcal{S}_{\rm eff}[\boldsymbol{\phi}] := (-1)\mathrm{Tr}\,\log\left[\partial_\tau + \sum_{j=1}^{d-1}(-\mathrm{i}\partial_j)\,\alpha_j + \sum_{\{\beta\}} 2\mathrm{i}\beta\,\phi_\beta\right] + \frac{1}{\lambda r}\sum_{\{\beta\}}\mathrm{Tr}\,(\phi_\beta^2).$$

Saddle point approximation

+ including fluctuations about the direction in which ϕ freezes

Nonlinear sigma model + topological term

Abanov, Wiegmann
Nucl. Phys. B (2000)

$$Z_{\rm bd} \approx \int \mathcal{D}[\boldsymbol{\phi}]\,\delta(\boldsymbol{\phi}^2 - 1)\,e^{-S_{\rm QNLSM} - S_{\rm top}},$$

$$S_{\rm QNLSM} = \frac{1}{2\,g}\int \mathrm{d}\tau \int \mathrm{d}^{d-1}\boldsymbol{x}\;(\partial_i\boldsymbol{\phi})^2$$

$\boldsymbol{\phi} \in S^{N(\nu)-1}$

Target space of NLSM is a sphere generated by N(ν) anticommuting dynamical masses β's

Topological term in NLSM

The presence or absence of a topological term is determined by the homotopy group of the target space.

$\pi_0(S^{N(\nu)-1}) \neq 0$ domain wall

$\pi_1(S^{N(\nu)-1}) \neq 0$ vortex

$\pi_d(S^{N(\nu)-1}) \neq 0$

$\pi_{d+1}(S^{N(\nu)-1}) \neq 0$ Wess-Zumino term

Nontrivial homotopy group

Topological term in NLSM

Boundary states remain gapless

Condition for the breakdown

$\pi_D(S^{N(\nu)-1}) = 0$ for $D = 0, \ldots, d+1$

Topological defects in the dynamical mass bind fermion zero-energy states.

$\nu_{\min}$: the minimum ν satisfying the above condition, $\mathbb{Z} \to \mathbb{Z}_{\nu_{\min}}$

Example: 3D class DIII (3He-B phase)

$\nu = 1$

Bulk: $\mathcal{H}^{(0)}(\boldsymbol{x}) := -\mathrm{i}\partial_1\, X_{31} - \mathrm{i}\partial_2\, X_{02} - \mathrm{i}\partial_3\, X_{11} + m(\boldsymbol{x})\, X_{03}.$

$\mathcal{T} := \mathrm{i}X_{20}\,\mathsf{K},$
$\mathcal{C} := X_{01}\,\mathsf{K}.$

Boundary: $\mathcal{H}^{(0)}_{\mathrm{bd}}(x,z) = -\mathrm{i}\partial_x \tau_3 - \mathrm{i}\partial_z \tau_1,$

$\mathcal{T}_{\mathrm{bd}\,\nu} := \mathrm{i}\tau_2 \otimes \mathbb{1}\,\mathsf{K}$
$\mathcal{C}_{\mathrm{bd}\,\nu} := \tau_0 \otimes \mathbb{1}\,\mathsf{K}.$

ν copies

Dynamical mass: $\tau_2 \otimes M(\tau, x, z)$

ν × ν Real symmetric matrix

Dynamical masses break T, but preserves C.

Space of masses (2D class D)
= real Grassmannians:

$R_0 = \bigcup_{k=1}^{\nu} O(\nu)/\left[O(k)\times O(\nu-k)\right]$

<u>ν=1:</u> $M = \pm 1$

<u>ν=2:</u> $M = X_1, X_3$

<u>ν=4:</u> $M = X_{13}, X_{33}, X_{01}$

<u>ν=8:</u> $M = X_{133}, X_{333}, X_{013}, X_{001}, X_{212}$

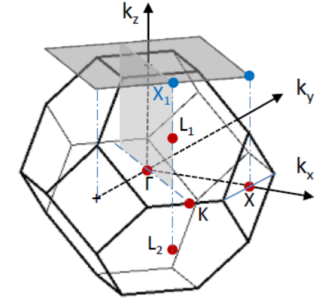
D	$\pi_D(R_0)$	ν	Topological obstruction
0	$\mathbb{Z}$	1	Domain wall
1	$\mathbb{Z}_2$	2	Vortex line
2	$\mathbb{Z}_2$	4	Monopole
3	0		
4	$\mathbb{Z}$	8	WZ term
5	0		
6	0		
7	0		
8	$\mathbb{Z}$	16	None

Higher dimensions

- Z_2 entries are stable.
- Z in even dimensions is stable.
- Z in odd dimensions is unstable.

Class	T	C	Γ_5	V_d	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$	$d=8$
A	0	0	0	C_{0+d}	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	0	0	1	C_{1+d}	$\mathbb{Z}_4$	0	$\mathbb{Z}_8$	0	$\mathbb{Z}_{16}$	0	$\mathbb{Z}_{32}$	0
AI	+1	0	0	R_{0-d}	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	+1	+1	1	R_{1-d}	$\mathbb{Z}_8$	0	0	0	$\mathbb{Z}_{16}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	0	+1	0	R_{2-d}	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
DIII	−1	+1	1	R_{3-d}	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{16}$	0	0	0	$\mathbb{Z}_{32}$	0
AII	−1	0	0	R_{4-d}	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
CII	−1	−1	1	R_{5-d}	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{16}$	0	0	0
C	0	−1	0	R_{6-d}	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
CI	+1	−1	1	R_{7-d}	0	0	$\mathbb{Z}_4$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{32}$	0

3D topological crystalline insulator (SnTe) (TRS+ reflection → Z classification)



Hsieh et al. 2012

Boundary: $\mathcal{H}^{(\mathrm{dyn})}_{\mathrm{bd}}(x,y) = -\mathrm{i}\partial_x\,\sigma_2 - \mathrm{i}\partial_y\,\sigma_1 + M(\tau,x,y)\,\sigma_3$

BdG: $\mathcal{H}^{(\mathrm{dyn})}_{\mathrm{bd}} = \left(-\mathrm{i}\partial_x\,\sigma_2\otimes\rho_3 - \mathrm{i}\partial_y\,\sigma_1\otimes\rho_0\right)\otimes \mathbb{1} + \gamma'(\tau,x,y),$

$R_0 = \bigcup^{\nu} O(\nu)/\left[O(k)\times O(\nu-k)\right]$

Minimal TCI
= 2copies of 3He-B phase →

D	$\pi_D(R_0)$	ν	Topological obstruction
0	$\mathbb{Z}$		
1	$\mathbb{Z}_2$	1	Vortex
2	$\mathbb{Z}_2$	2	Monopole
3	0		
4	$\mathbb{Z}$	4	WZ term
5	0		
6	0		
7	0		
8	$\mathbb{Z}$	8	None

$\mathbb{Z} \to \mathbb{Z}_8$

Summary and outlook

- Classification of topological crystalline insulators
  - General spatial symmetry that cannot fit into Clifford algebras? Twisted equivariant K-theory?

  T. Morimoto, A.Furusaki, Phys. Rev. B 88, 125129 (2013).

- Effects of interactions on topological insulators
  - Nonlinear sigma model analysis over spherical target spaces
  - More general interactions?
  - New topological phases that emerges with interactions?

  T. Morimoto, A.Furusaki, C.Mudry, Phys. Rev. B 92, 125104 (2015)